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* Every convergent sequence is a Cauchy sequence. But every Cauchy sequence is not necessarily a convergent sequence.

We show it by an example.

Let $X = R - \{0\}$ and d be the usual metric for R given by $d(x,y) = |x-y|$.

Consider the sequence defined by $x_n = \frac{1}{n}$, $n \in N$

We show that $\{x_n\}$ is a cauchy sequence.

For each $\epsilon > 0$, let us choose a (+)ve integer $n_0 > \frac{2}{\epsilon}$

If m, n are (+)ve integers and $m, n > n_0$

then $m > n_0 \Rightarrow m > \frac{2}{\epsilon} \Rightarrow \frac{1}{m} < \frac{\epsilon}{2}$

Similarly $\frac{1}{n} < \frac{\epsilon}{2}$.

$\Rightarrow \frac{1}{m} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Now $d(x_m, x_n) = |x_m - x_n|$ by definition of d

$\leq |x_m| + |x_n| = \frac{1}{m} + \frac{1}{n} < \epsilon$

$\Rightarrow d(x_m, x_n) < \epsilon \Rightarrow \{x_n\}$ is a cauchy sequence.

But limit of $\{x_n\} = 0$ which is not in X. $\Rightarrow$ $\{x_n\}$ is not conv.

Q If a convergent sequence in a metric Space has infinitely many distinct points then its limit is a limit point of the set of points of the sequence.

Proof Let (X,d) be a metric space.

Let $\{x_n\}$ be a convergent sequence in X.

Let its limit be x.

Let x is not a limit point of the set of points of the sequence.

$\Rightarrow$ $\exists$ an open sphere $S_r(x)$ with centre at x which contains no point of the sequence different from x.

But x is the limit of the sequence.

$\Rightarrow$ all x_n's from some place on must lie in $S_r(x)$

i.e. they must coincide with x.

$\Rightarrow$ there are only finitely many distinct points in the sequence.

Hence the theorem